Thus

$$\frac{Nu_x}{Nu_x^K} = \frac{(-\,d\theta/dy|_w)}{(-\,k\theta/dy|_w)^K} + \frac{3}{4}\epsilon_w P\left(\frac{\tau_x}{Nu_x^K}\right)\left(1 - \frac{5}{4}\frac{\tau_x}{Nu_x^K}\right)$$

and the local thermal entropy production on the wall is

$$s_x''' = -\frac{1}{T_w^2}(q_w^K + q_w^R)\left(\frac{\partial T}{\partial y}\right)\bigg|_w \tag{31}$$

Introducing a wall local entropy production number, $\Pi_x = s_x''' x^2/k$, Eq. (31) may be arranged as

$$\Pi_x = \left(1 - \frac{T_\infty}{T_w}\right)^2\left(1 + \frac{q_w^R}{q_w^K}\right)\left[\frac{(\partial T/\partial y)|_w}{(T_w - T_\infty)/x}\right]^2 \tag{32}$$

With the definition of local Nusselt number

$$Nu_x = \frac{q_x^C}{q_w^K} = \frac{q_w^K}{q_x^K} = \frac{(\partial T/\partial y)|_w}{(T_w - T_\infty)/x} \tag{33}$$

Eq. (32) may finally be expressed as

$$\Pi_x = \left(1 - \frac{T_\infty}{T_w}\right)^2\left(1 + \frac{q_w^R}{q_w^K}\right)Nu_x^2 \tag{34}$$

Concluding Remarks

The radiation-affected forced convection over a flat plate is investigated in terms of thin gas. The distribution of entropy production within and outside the radiation-affected thermal boundary layer is evaluated. The retained nonlinearity of temperature in the entropy production leads to an extremum in this production within the boundary layer rather than on the boundary.

References

[1]Bejan, A., *Entropy Generation through Heat and Fluid Flow,* Wiley, New York, 1982.

[2]Bejan, A., *Advanced Enginering Thermodynamics,* Wiley-Interscience, New York, 1988.

[3]Arpaci, V. S., "Radiative Entropy Production," *AIAA Journal,* Vol. 24, Nov. 1986, pp. 1859-1860.

[4]Arpaci, V. S., "Radiative Entropy Production – Lost Heat into Entropy," *International Journal of Heat and Mass Transfer,* Vol. 30, Oct. 1987, pp. 2115-2123.

[5]Arpaci, V. S. and Selamet, A., "Radiative Entropy Production," *Proceedings of 8th International Heat Transfer Conference,* Vol. 2, 1986, pp. 729-734.

[6]Arpaci, V. S. and Selament, A., "Entropy Production in Flames," *Combustion and Flame,* Vol. 73, Sept. 1988, pp. 251-259.

[7]Curle, N., *The Laminar Boundary Layer Equations,* Clarendon, Oxford, 1962.

[8]Arpaci, V. S. and Larsen, P. S., *Convection Heat Transfer,* Prentice-Hall, Englewood Cliffs, NJ, 1984, Chaps. 4 and 5.

[9]Lord, H. A. and Arpaci, V. S., "Effect of Nongray Thermal Radiation on Laminar Forced Convection over a Heated Horizontal Plate," *International Journal of Heat and Mass Transfer,* Vol. 13, Nov. 1970, p. 1737-1751.

[10]Lentini, M. and Pereyra, V., "An Adaptive Finite Difference Solver for Nonlinear Two Point Boundary Problems with Mild Boundary Layers," *Journal of Numerical Analysis*, Vol. 14, 1978, pp. 91-111; also Informational Mathematical and Statistical Libraries manuals, Vol. 1, code: DVCPR, 1982, Chap. D.

[11]Hull, T. E., Enright, W. H., and Jackson, K. R., "User's Guide for DVERK – a Subroutine for Solving Non-stiff ODE's," Dept. of Computer Science, Univ. of Toronto, Toronto, Canada, *TR No. 100,* 1976; also IMSL manuals, Vol. 1, code: DVERK, 1982, Chap. D.

[12]Özişik, M. N., *Radiative Transfer,* Wiley-Interscience, New York, 1973, Chap. 9.

[13]Siegel, R. and Howell, J. R., *Thermal Radiation Heat Transfer,* 2nd ed., Hemisphere, Washington, DC, 1981, Chap. 14.

[14]Sparrow, E. M. and Cess, R. D., *Radiation Heat Transfer,* Hemisphere, Washington, DC, 1978, Chaps. 7 and 10.

Attenuating Thin Gas

V. S. Arpaci*
University of Michigan, Ann Arbor, Michigan
and
J. S. Troy†
McDonnell Douglas Space Systems Company, Huntington Beach, California

Introduction

A DERIVATION of the monochromatic intensity balance (transfer equation) under the influence of emission, absorption, and scattering is available in the literature.[1,2] The following brief review is for later convenience.

The monochromatic transfer equation integrated over the frequency domain gives

$$l_j\frac{\partial I}{\partial x_j} = \kappa I_o + \frac{\sigma_s}{4\pi}\int_{\Omega'} P(l_i',l_i)I(l_i')\,d\Omega' - \beta I \tag{1}$$

where I is the intensity, I_o its equilibrium state, κ the absorption coefficient, σ_s the scattering coefficient, $\beta = \kappa + \sigma_s$ the extinction coefficient, Ω the solid angle, and $P(l_i',l_i)$ the phase function that satisfies

$$\frac{1}{4\pi}\int_{\Omega'} P(l_i',l_i)\,d\Omega' = 1 \tag{2}$$

l_i being the direction of the optical energy balance and l_i' the direction of the scattering.

The first specular moment of Eq. (1) yields the radiative energy balance

$$\frac{\partial q_j^R}{\partial x_j} = 4\kappa E_b + \frac{\sigma_s}{4\pi}\int_\Omega\int_{\Omega'} P(l_i',l_i)I\,d\Omega'\,d\Omega - \beta J \tag{3}$$

where $q_i^R = \int_\Omega I l_i\,d\Omega$ is the radiative heat flux in the x_i direction, $E_b = \pi I_o$ the equilibrium blackbody emissive power, and $J = \int_\Omega I\,d\Omega$ the specular integrated intensity. In view of Eq. (2) and

$$\int_\Omega\int_{\Omega'} P(l_i',l_i)I\,d\Omega' d\Omega = \int_{\Omega'}\left[\int_\Omega P(l_i',l_i)I\,d\Omega\right]d\Omega' = 4\pi J \tag{4}$$

Eq. (3) may be rearranged as

$$\frac{\partial q_j^R}{\partial x_j} = \kappa(4E_b - J) \tag{5}$$

The second specular moment of the transfer equation leads to the radiative momentum balance

$$\frac{\partial \Pi_{ij}}{\partial x_j} = \frac{\sigma_s}{4\pi}\int_\Omega\int_{\Omega'} P(l_j',l_j)Il_i\,d\Omega'\,d\Omega - \beta q_i^R \tag{6}$$

where Π_{ij} is related to the radiative stress τ_{ij}^R by

$$\tau_{ij}^R = \frac{1}{c}\int_\Omega I l_i l_j\,d\Omega = \frac{1}{c}\Pi_{ij} \tag{7}$$

c being the velocity of light.

Received Jan. 30, 1989; revision received Aug. 9, 1989.

*Professor, Department of Mechanical Engineering and Applied Mechanics.

†Senior Engineer/Scientist, Advanced Technology Center.

In view of the nature of this study, consider a weighted anisotropy for scattering:

$$\alpha = \frac{\int_\Omega \int_{\Omega'} P(l_i',l_i) I l_i \, \mathrm{d}\Omega' \, \mathrm{d}\Omega}{4\pi \int_\Omega I l_i \, \mathrm{d}\Omega} \tag{8}$$

In terms of this anisotropy, Eq. (6) may be written as

$$\frac{\partial \Pi_{ij}}{\partial x_j} = (\alpha\sigma_s - \beta) q_i^R \tag{9}$$

where $\alpha = 0$ corresponds to isotropic scattering and $0 \leq \alpha \leq 1$ to a degree of weighted anisotropic scattering. In terms of the albedo of single scattering, $\omega_s = \sigma_s/\beta$, Eq. (9) yields

$$q_i^R = -\frac{1}{\omega_o \kappa} \frac{\partial \Pi_{ij}}{\partial x_j} \tag{10}$$

where

$$\omega_o = \frac{1 - \alpha\omega_s}{1 - \omega_s}$$

is a parameter characterizing scattering. The complete determination of J, q_i^R, and Π_{ij} requires another relation (the closure problem). An approximation for this relation is the assumption of local thermodynamic equilibrium under which the radiative stress is replaced by the radiative pressure p^R,

$$\tau_{ij}^R = p^R \delta_{ij} = \frac{1}{3c} J$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Thus, in view of Eq. (7),

$$\Pi_{ij} = \frac{1}{3} J \delta_{ij} \tag{11}$$

Then, the second moment in terms of Eq. (11) becomes

$$q_i^R = -\frac{1}{3\omega_o \kappa} \frac{\partial J}{\partial x_i} \tag{12}$$

The combination of Eqs. (5) and (12) and elimination of J result in the radiative energy balance

$$(\nabla^2 - 3\omega_o \kappa^2) q_i^R = 4\kappa \frac{\partial E_b}{\partial x_i} \tag{13}$$

For small values of κ, in line with neglecting the term involving κ^2, Eq. (13) reduces to its optically thin gas limit

$$\frac{\partial q_j^R}{\partial x_j} = 4\kappa E_b \tag{14}$$

In this equation, κ corresponds to the Planck mean absorption coefficient κ_P, defined by

$$\kappa_P E_b = \int_0^\infty \kappa_\nu E_{b\nu} \, \mathrm{d}\nu \tag{15}$$

where ν is the frequency. Similarly, for large κ, Eq. (13) reduces to its optically thick gas limit

$$q_i^R = -\frac{1}{3\omega_o \kappa} \frac{\partial E_b}{\partial x_i} \tag{16}$$

In this equation, κ corresponds to the Rosseland mean absorption coefficient κ_R, defined by

$$\frac{1}{\kappa_R} \frac{\partial E_b}{\partial x_i} = \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial E_{b\nu}}{\partial x_i} \, \mathrm{d}\nu \tag{17}$$

In terms of the mean absorption coefficient $\kappa_M = (\kappa_P \kappa_R)^{1/2}$ and the weighted nongrayness $\eta = (\kappa_P/\kappa_R)^{1/2}$, the radiative constitution becomes

$$(\nabla^2 - 3\omega_o \kappa_M^2) q_i^R = 4\eta\kappa_M \frac{\partial E_b}{\partial x_i} \tag{18}$$

The incorporation of κ_M and η into the formulation for arbitrary optical thickness can be found in Traugott,[3] Cogley et al.,[4] Arpaci and Gozum,[5] and Arpaci and Bayazitoglu.[6]

Intermediate Behavior of the Attenuating Thin Gas

The optically thin and optically thick limits are simplifications based, respectively, on small and large optical thicknesses of the gas. The Rosseland approximation reduces radiative transport in optically thick media to a diffusion problem, and is developed in conjunction with astrophysical problems, where boundaries either are absent or are not of prime concern. An improved thick gas model including the effect of boundaries was developed by Arpaci[7] and Arpaci and Larsen.[8] The thin gas approximation was developed by Cess,[9] in connection with boundary layer problems. However, both limits follow lengthy mathematical approximation of some integral exponentials. An intuitive physical approach leading to a simple order-of-magnitude shorter development than approaches in the literature escaped the attention of previous studies. This intuitive approach is the objective of the present paper.

The homogeneous solution to Eq. (18), in terms of $\kappa_E = \sqrt{\omega_o}\kappa_M$, is

$$q_x^R = b_0 e^{-\sqrt{3}\kappa_E x} \tag{19}$$

for the radiative heat flux and

$$\left.\frac{\mathrm{d}q_x^R}{\mathrm{d}x}\right|_H = b_1 e^{-\sqrt{3}\kappa_E x} \tag{20}$$

for its gradient.

The nonhomogeneous solution for an optically thin gas, obtained by neglecting the term involving κ_M^2 in Eq. (18), leads to

$$\left.\frac{\mathrm{d}q_x^R}{\mathrm{d}x}\right|_{NH} = 4\eta\kappa_M E_b = b_2 \tag{21}$$

The complete solution, obtained from the combination of Eqs. (20) and (21), is

$$\frac{\mathrm{d}q_x^R}{\mathrm{d}x} = 4\eta\kappa_M E_b + b_2 + b_1 e^{-\sqrt{3}\kappa_E x} \tag{22}$$

Noting that as $x \to \infty$, $E_b \to E_{b\infty}$, and $\mathrm{d}q_x^R/\mathrm{d}x \to 0$, one finds that $b_2 = -4\eta\kappa_M E_{b\infty}$ and

$$\frac{\mathrm{d}q_x^R}{\mathrm{d}x} = 4\eta\kappa_M (E_b - E_{b\infty}) + b_1 e^{-\sqrt{3}\kappa_E x} \tag{23}$$

Near boundaries, the intermediate gas flux reduces to the well-known thin gas flux,[10]

$$\frac{\mathrm{d}q_x^R}{\mathrm{d}x} = 4\eta\kappa_M \left[(E_b - E_{b\infty}) - \frac{\epsilon_w}{2} (E_{bw} - E_{b\infty}) \right] \tag{24}$$

where E_{bw} is the wall emissive power and ϵ_w is the wall emissivity. Evaluating Eq. (23) on the wall and equating to Eq. (24), one finds that

$$b_1 = -4\eta\kappa_M \frac{\epsilon_w}{2} (E_{bw} - E_{b\infty})$$

and, subsequently, the constitutive relation for q_x^R is

$$\frac{\mathrm{d}q_x^R}{\mathrm{d}x} = 4\eta\kappa_M \left[(E_b - E_{b\infty}) - \frac{\epsilon_w}{2} (E_{bw} - E_{b\infty}) e^{-\sqrt{3}\kappa_E x} \right] \tag{25}$$

where the first emissive power difference shows the effect of the gas and the second emissive power difference denotes the attenuating wall effect discussed in Troy.[11] A special case of this result for negligible scattering has been obtained by Lord and Arpaci,[12] following an elaborate mathematical development based on an approximation of the integral formulation of radiation. Far from boundaries, the thin gas flux [Eq. (24)] leads to an erroneous finite flux and is not suitable for studies involving semi-infinite geometry.

In the absence of conduction, there exists a temperature jump between the wall and the gas immediately next to he wall. The radiative heat flux (or its gradient) is linearly related to the emissive power [see, for example, Eq. (18)], and the principle of superposition applies in terms of E_b. Therefore, one can write the heat flux gradient, for the thin gas, as

$$\frac{dq_x^R}{dx} = \left(\frac{dq_x^R}{dx}\right)_{\text{gas}} + \left(\frac{dq_x^R}{dx}\right)_{\text{jump}} + \left(\frac{dq_x^R}{dx}\right)_{\text{gas at wall}} \tag{26}$$

in which $(dq_x^R/dx)_{\text{gas}}$ denotes the contribution to the difference in radiation between each local element of gas and the gas at infinity, $(dq_x^R/dx)_{\text{jump}}$ arises from the temperature difference associated with the jump, and $(dq_x^R/dx)_{\text{gas at wall}}$ arises from the temperature difference between the gas at the wall and the gas far from the wall. It can be shown in a manner similar to the development leading to Eq. (25) that

$$\frac{dq_x^R}{dx} = 4\eta\kappa_M\left\{(E_b - E_{b\infty}) - \frac{\epsilon_w}{2}\left[\left(E_{bw} - E_b(0)\right) + \left(E_b(0) - E_{b\infty}\right)\right]e^{-\sqrt{3}\kappa_E x}\right\} \tag{27}$$

where $E_b(0)$ is the emissive power of the gas at the wall, differing from E_{bw} in the absence of conduction.

Dimensional Arguments

The upper bound of the attenuating thin gas approximation can be determined by comparing it with the exact radiative heat flux. The exact flux for cases with negligible scattering, obtained by Spiegel[13] from a mathematical study, is

$$\frac{q^R}{4\eta E_b} = \tau(1 - \tau \cot^{-1}\tau) \tag{28}$$

where $\tau = \kappa_M L$ is the optical thickness and L is a characteristic length. A first-order representation of Eq. (18) by Arpaci[14] yields the Eddington approximation

$$\frac{q^R}{4\eta E_b} \sim \frac{\tau}{1 + 3\tau_E^2} \tag{29}$$

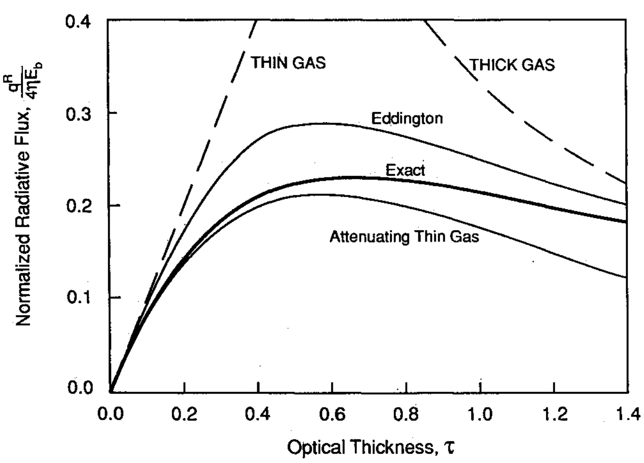

Fig. 1 Comparison of radiative heat fluxes.

where $\tau_E = \sqrt{\omega_o}\tau$. For $\tau \to 0$, $q^R/4\eta E_b \sim \tau$, which is the classical thin gas or emission-dominated limit. For $\tau \to \infty$, Eqs. (28) and (29) reduce to the thick gas limit, $q^R/4\eta E_b \sim 1/3\omega_o\tau$. For steady one-dimensional pure radiation, the energy balance reduces to

$$\frac{dq_x^R}{dx} = 0$$

which, from Eq. (25), gives

$$E_b - E_{b\infty} = \frac{\epsilon_w}{2}(E_{bw} - E_{b\infty})e^{-\sqrt{3}\tau_E} \tag{30}$$

For steady one-dimensional radiation and conduction, Eq. (30) continues to be valid outside of the conductive boundary layer. Accordingly, the emissive power decays exponentially with optical thickness; that is, $E_b - E_{b\infty} \sim \exp(-\sqrt{3}\tau_E)$ and the first order representation of Eq. (25) results in

$$\frac{q^R}{4\eta E_b} \sim \tau e^{-\sqrt{3}\tau_E} \tag{31}$$

The preceding expressions are plotted as a function of optical thickness in Fig. 1 for cases with negligible scattering ($\tau_E = \tau$). The attenuating thin gas flux follows the exact flux more closely than the flux associated with the Eddington approximation, because the former uses the Eddington approximation only in the attenuation term. Ref. 14 shows that a maximum of 29% difference exists in the neighborhood of $\tau = 0.4$ between the Eddington approximation and the exact representation. The error between the attenuating thin gas flux and the exact flux does not reach 29% until about $\tau \simeq 1$. In the thick optical range $\tau > 1$, the quantitative difference between the present model and the exact flux is due to the appreciable effect of "radiative diffusion," which is neglected from the emission component in the model.

References

[1]Chandrasekhar, S., *Radiative Transfer,* Dover, New York, 1960.

[2]Özişik, M. N., *Radiative Transfer,* Wiley-Interscience, New York, 1973.

[3]Traugott, S. C., "Radiative Heat-Flux Potential for a Nongrey Gas," *AIAA Journal,* Vol. 4, No. 3, 1966, p. 541-542.

[4]Cogley, A. C., Vincenti, W. G., and Gilles, S. E., "Differential Approximation for Radiative Transfer in a Nongrey Gas near Equilibrium," *AIAA Journal,* Vol. 6, No. 3, 1968, p. 551-533.

[5]Arpaci, V. S., and Gozum, D., "Thermal Stability of Radiating Fluids: The Benard Problem," *Physics of Fluids,* Vol. 16, 1973, p. 581-588.

[6]Arpaci, V. S., and Bayazitoglu, Y., "Thermal Stability of Radiating Fluids: Asymmetric Slot Problem," *Physics of Fluids,* Vol. 16, 1973, p. 589-593.

[7]Arpaci, V. S., "Effect of Thermal Radiation on the Laminar Free Convection From a Heated Vertical Plate," *International Journal of Heat and Mass Transfer,* Vol. 11, 1968, p. 871-881.

[8]Arpaci, V. S., and Larsen, P. S., "A Thick Gas Model Near Boundaries," *AIAA Journal,* Vol. 4, 1969, p. 602-606.

[9]Cess, R. D., "Radiation Effects upon Boundary-Layer Flow of an Absorbing Gas," *Journal of Heat Transfer,* Vol. C86, 1964, p. 469-475.

[10]Sparrow, E. M., and Cess, R. D., *Radiation Heat Transfer,* McGraw-Hill, Washington, DC, 1978.

[11]Troy, J. S., "Radiating Gas Acoustics," Ph.D. Thesis, Dept. of Mechanical Engineering and Applied Mechanics, Univ. of Michigan, Ann Arbor, MI, 1986.

[12]Lord, H. A., and Arpaci, V. S., "Effect of Nongray Thermal Radiation on Laminar Forced Convection over a Heated Horizontal Plate," *International Journal of Heat and Mass Transfer,* Vol. 13, 1970, p. 1737-1751.

[13]Spiegel, E. A., "The Smoothing of Temperature Fluctuations by Radiative Transfer," *Astrophysics Journal,* Vol. 126, 1957, pp. 202-207.

[14]Arpaci, V. S., "Hookean and Stokesean Implications of the Radiative Stress," American Society of Mechanical Engineers, HTD-40, 1984.